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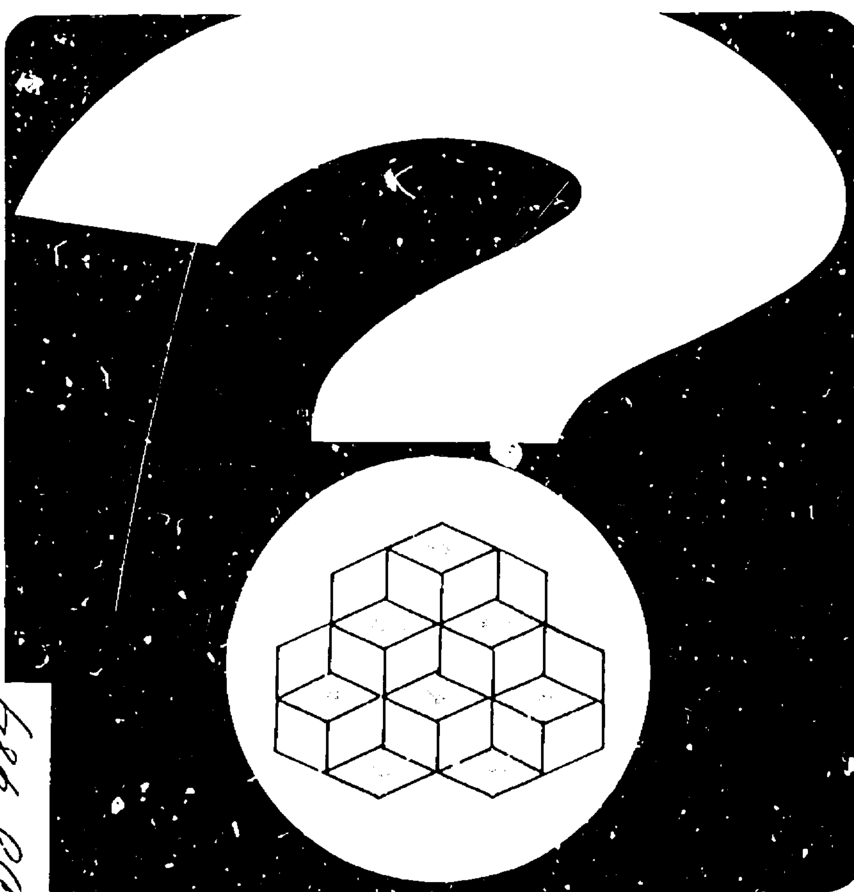
ABSTRACT

This is the sixth in a series of ten self-contained units designed for use by students in ninth grade general mathematics classes. This unit is divided into eight sections dealing with different types of mathematical thinking. Some topics presented include functions, permutations and combinations, symmetry, inductive thinking, and logic. Though the topics are standard they are dealt with in non-traditional methods emphasizing discovery learning. Included are many diagrams, exercises, and topics for discussion. (CT)

6 Mathematical Thinking

EXPERIENCES IN MATHEMATICAL DISCOVERY



National Council of
Teachers of Mathematics

UNIT SIX OF

Experiences in Mathematical Discovery

Mathematical Thinking

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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Preface

"Experiences in Mathematical Discovery" is a series of ten self-contained units, each of which is designed for use by students of ninth-grade general mathematics. These booklets are the culmination of work undertaken as part of the General Mathematics Writing Project of the National Council of Teachers of Mathematics (NCTM).

The titles in the series are as follows:

Unit 1: *Formulas, Graphs, and Patterns*

Unit 2: *Properties of Operations with Numbers*

Unit 3: *Mathematical Sentences*

Unit 4: *Geometry*

Unit 5: *Arrangements and Selections*

Unit 6: *Mathematical Thinking*

Unit 7: *Rational Numbers*

Unit 8: *Decimals, Ratios, Percents*

Unit 9: *Positive and Negative Numbers*

Unit 10: *Measurement*

This project is experimental. Teachers may use as many units as suit their purposes. Authors are encouraged to develop similar approaches to the topics treated here, and to other topics, since the aim of the NCTM in making these units available is to stimulate the development of special materials that can be effectively used with students of general mathematics.

Preliminary versions of the units were produced by a writing team that met at the University of Oregon during the summer of 1963. The units were subsequently tried out in ninth-grade general mathematics classes throughout the United States.

Oscar F. Schaaf, of the University of Oregon, was director of the 1963 summer writing team that produced the preliminary materials. The work of planning the content of the various units was undertaken by Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Oklahoma; Paul S. Jorgensen, Carleton College, Northfield, Minnesota; Kenneth P. Kidd, University of Florida, Gainesville, Florida; and Max Peters, George W. Wingate High School, Brooklyn, New York.

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Finally, a word of grateful thanks is extended to the NCTM headquarters staff for their assistance, and in particular to Clare Stifft and Gladys Majette for their careful attention to production editing details.

EMIL J. BERGER
Chairman, Advisory Committee
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Contents

Experiences in Mathematical Discovery

Mathematical Thinking

1

Don and Phil were walking home from school one day during the month of November. Don said, "I'm glad winter is almost here. I think winter is the best time of the year." To this Phil replied, "Oh no! I think summer is the best time of the year." For the rest of the way home Don tried to convince Phil that winter is the best time of the year, and Phil tried to convince Don that summer is.

Class Discussion

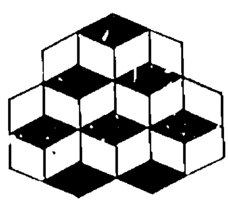
1. One of the two boys in the above story likes to ski and ice skate. The other is the star third baseman on an American Legion baseball team. Which one is Phil and which one is Don?

2. What is the "best" time of the year for a man who likes to play golf; for a boy who is a Green Bay Packers' fan; for a girl from Minnesota who likes to swim?

3. What is the "worst" time of the year for a resident of Fairbanks, Alaska; for an American citizen who has to pay income taxes; for a department store salesclerk who gets paid by the hour; for a mailman?

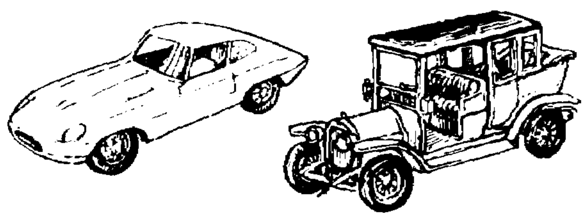
4. Sally and Maureen were listening to records one afternoon. Sally said, "I think the Four Tunes are the greatest singing group around." "Really?" said Maureen. "I think the Three Wise Men are the most. The Three Wise Men are really cute and they have the neatest haircuts." "That doesn't mean a thing," replied Sally. "What matters is the message in their songs, and the Four Tunes really turn me on."
   a. Do you think Sally could convince Maureen that the Four Tunes are a better singing group than the Three Wise Men?
   b. Do you think Sally's definition of "the greatest" and Maureen's definition of "the most" are the same?
   c. What does Maureen mean by "cute"? By "neatest"?
   d. What does Sally mean by "the message"?

5. Ted was having an argument with his sister Jo. Ted thought there were six cubes in the picture at the right, but Jo said there were seven. Who was right? Before you answer, turn this page upside down and look at the picture again.



6. Mr. Pratto and his son Marty were looking at cars in a used car lot. "Now there is a good buy," said Mr. Pratto looking at a 1962 Dodge. "Gosh, Dad! How can you say such a thing?" answered Marty. "Well," said Mr. Pratto, "It is big and roomy and should be comfortable. It has power steering and power brakes, and the six-cylinder engine should give good gas mileage." "Ugh!" was Marty's comment.

a. What kind of used car do you think Marty would consider a "good buy"?



"E" Type Jaguar 1910 Benz Limousine

b. If you were a used car salesman, what kind of car would you show a teenager; a newly married couple with no children; a traveling salesman; a retired railroad engineer?

Summary—1

Differences in points of view among people depend on the way in which they have been raised, how much money they have, their ages, what part of the country they come from, and many other things.

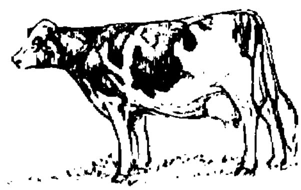
In the exercises above you were asked to give your reactions to certain situations while looking at them from someone else's point of view. Many of the questions asked did not have definite answers. One reason for this is that the discussions centered around words that could have different meanings for different people. If a discussion is to result in a definite conclusion, it is necessary to agree on the meanings of the words and symbols that are used.

Exercises—1

1. For each statement, tell at least one thing on which two people would have to agree in order to decide whether or not the statement is true.

a. A test score of 55 is low
b. John F. Kennedy was a good president.
c. Colorado is the prettiest state in the United States.
d. Driving faster than 60 miles per hour is dangerous.
e. Failure to vote in an election is unpatriotic.

2. For each exercise give at least one reason that might explain why the people named have the opinions they do.
   a. Mr. Simmons thinks Washington, D.C., is the best place in the United States in which to live.
   b. Martha's mother thinks hot oatmeal makes a good breakfast.
   c. Walter thinks social studies is the hardest subject in school.
   d. Farmer Stein feels that Holstein cows are the best kind of cows to raise.



   e. Mr. Corey says it is more sensible to buy a new car every year than to keep one car for several years.
   f. Belinda claims long division is easy. Her sister, Bertha, claims it is very hard.
   g. Mr. Windom thinks it is smarter to travel to Europe by airplane than by boat. Mrs. Windom thinks the opposite.
   h. Ronald's teacher says Indians are the only true Americans.

3. There is a word or phrase in each statement in exercise 2 that could have different meanings for different people. Identify this word or phrase. Then tell what this word or phrase might mean to the people involved.

In this section you saw that different people can have different points of view about certain things. In the sections that follow you will see how people use ideas from mathematics to form opinions.

2 A Game of Tennis

Sam and Steve were playing tennis, but neither seemed to be playing very well. After a while Sam said "I wonder what is wrong? Maybe the ball we are using is no good." Steve replied, "Why don't we test it and find out? We can drop the ball from different heights and see how high it bounces." Luckily the boys were playing on a tennis court near their school. So Sam ran into the woodshop and borrowed a measuring tape. The results of the boys' experiment are shown below.

Experiment 1

Distance Dropped (Inches)	12	18	24	30	36	42
Height of Bounce (Inches)	4	5	8	10	$12\frac{1}{2}$	13

"There does not seem to be any pattern in the results," said Steve. "The ball must be defective."

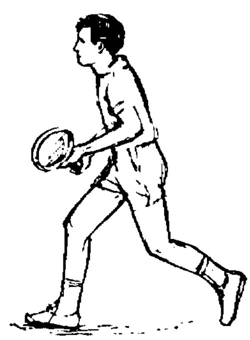
"Our results don't prove a thing," replied Sam. "Maybe all tennis balls behave like the one we have. Let's repeat the experiment with another ball that we know is good." Steve dashed into the coach's office and borrowed a brand-new tennis ball. When the boys repeated the experiment with the new ball, they got the results below.

Experiment 2

Distance Dropped (Inches)	12	18	24	30	36	42
Height of Bounce (Inches)	6	$9\frac{1}{2}$	$12\frac{1}{4}$	$14\frac{3}{4}$	$18\frac{1}{2}$	$20\frac{3}{4}$

"See! I told you so," shouted Steve. "The ball we were using is no good."

Class Discussion 2



1. Do you think Steve's conclusion is a sensible one?
2. What pattern do you think Steve saw in the second experiment? Can you state this as a rule?
3. Does the rule you stated fit each trial in the second experiment? If not, why not?
4. Do the results in the first experiment come close to fitting the rule you stated?
5. Can you think of another way to carry out an experiment to see if the tennis ball the boys were using is defective? What difficulties might you run into if you tried your method?

Sam and Steve conducted an experiment to arrive at a conclusion. They recorded the results of their experiment in a table. Then they examined the table for a mathematical relationship between the pairs of numbers in the table. Steve felt the relationship was so noticeable that he could form a rule about the pairs of numbers. He thought the rule proved his hunch about the first tennis ball.

1. **a.** In the table below, the second number in each column is two times the first number. Complete the table.

First Number	1	2	3	5	8	—	—
Second Number	2	4	6	—	—	20	30

   b. Write a formula that expresses the relationship in the table. Use f to represent the first number in each column, and s to represent the second number.

2. In the table below, the numbers represented by a and b are related by the formula $b = 3a - 1$. Complete the table.

a	1	2	3	5	8	11	15	—	—
b	2	5	8	—	—	—	—	17	29

3. A formula that relates the numbers represented by x and y is $y = \frac{1}{2}x + 3$. Complete the table.

x	1	2	3	—	10	—	15	—	—
y	$3\frac{1}{2}$	4	—	6	—	$8\frac{1}{2}$	—	14	15

4. The table below relates the length of the side of a square with the area of the square. Complete the table.

Length of Side	2	3	4	—	—	10	12
Area of Square	4	9	—	25	49	100	—

5. Let A represent the area of a square and let s represent the side of a square. Write a formula that relates A with s.

6. Write a formula that expresses the relationship between the numbers represented by x and y in the table below.

x	1	2	3	5	8	10
y	3	6	9	15	24	30

7. Write a formula that expresses the relationship between the numbers represented by m and n in the table below.

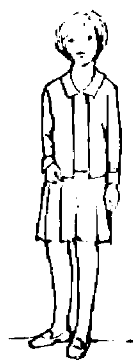
m	1	2	3	5	8	10
n	1	3	5	9	15	19

8. Complete the following table. Your answers should fit a formula that relates the given pairs of numbers. Can you tell what the formula is?

x	1	2	3	4	5	6	—	—	12	—
y	8	11	14	17	—	—	32	35	—	50

3 Nothing to Wear

One day in late August, Monica said to her father, "Dad, since I am going into junior high this fall, I think I should get some new clothes. I really don't have a thing to wear." Her father, Mr. Burkett, thought to himself, "Where have I heard this before?" But he answered patiently, "Monica, I am a carpenter, and I spend a lot of time figuring out different ways to make things. While you've been talking, I've been figuring out how many different outfits you have to wear. Not even counting your dresses, I figure that with your five different skirts and six different sweaters you can come up with thirty different outfits. So how can you say that you don't have a thing to wear?"



Class Discussion 3

1. Suppose Monica had four different skirts and seven different sweaters. How many different outfits could she wear?
2. From the practical point of view, what is wrong with Mr. Burkett's reasoning? What word probably has a different meaning for Monica than it does for Mr. Burkett?
3. Suppose Monica is an "oddball" about clothing and doesn't care whether or not colors clash. How many outfits would she have if she had three skirts and four sweaters?
4. If an outfit consists of a skirt, a blouse, and a sweater, how many different outfits could Monica wear if she had three different skirts, eight different blouses, and two different

sweaters? If she had four different skirts, seven different blouses, and five different sweaters?

5. Suppose Monica is not an "oddball" and does care whether or not skirts and sweaters match. Describe a method of figuring out how many different outfits she could wear that are harmonious in color. Try to invent a method that is better than just counting.

Summary—3

Mr. Burkett used a special method of counting to find the number of combinations of skirts and sweaters that Monica can wear. However, his definition of "outfit" is probably not the same as Monica's. Therefore, he came to a conclusion that is not practical as far as Monica is concerned. Mathematics is useful only when it is supported by clear thinking.

Exercises—3

1. Suppose again that Monica really is an "oddball" about clothing. Complete the table by giving the number of outfits she can wear.

Number of Skirts	Number of Sweaters	Total Number of Outfits
5	6	30
4	7	—
3	4	—
4	4	—

2. Tim has three suits and five ties in his wardrobe. Let F represent the first suit, S the second, and T the third. Also, let each of the numerals 1, 2, 3, 4, and 5 represent one of the five ties.

By listing first a suit and then a tie, indicate all suit-and-tie combinations. (For example, let S–3 represent the combination made up of the second suit and the third tie.)

3. How many different suit-and-tie combinations did you obtain in exercise 2? Could you have found this answer without actually counting all combinations?

4. Mr. Nielsen is planning a business trip from Denver to New York with a stopover in Chicago. He can go from Denver to Chicago on any one of four different flights and from Chicago to New York on any one of three different flights. In how many different ways can Mr. Nielsen make the trip from Denver to New York?

5. Suppose Mr. Nielsen had business in Boston. He can get from New York to Boston on either of two different trains. In how many different ways can Mr. Nielsen make the trip from Denver to Boston?

6. A battery on a baseball team consists of a pitcher and a catcher. If a team has six pitchers and two catchers, in how many different ways can a battery be chosen?

7. The letters A and B can be written in two different orders, AB and BA. The letters A, B, and C can be written in six different orders. ABC and CAB are two of the six different orders. Write down the four other possible orders.

8. In how many different orders can the four letters, A, B, C, and D be written? (Hint: Consider all possible orders with A as the first letter. Then consider all possible orders with B as the first letter, and so on.)

9. The information in exercise 7 and the answer to exercise 8 are recorded in the table at the top of the next page.

Number of Different Letters	Number of Different Orders
1	1
2	2
3	6
4	24
5	—

a. Did you get 24 for your answer to exercise 8?

b. Complete the table for five different letters. Do not try to get the answer by writing down all possible orders of five letters. There are over 100 of them! Instead, examine the table to find a pattern, and then use the pattern.

10. If a coin is tossed in the air, there are two possible results. The coin can land "heads" or "tails." Ordinarily we use H to represent "heads" and T to represent "tails." If the coin is tossed twice, there are four possible results. One of these can be represented by TH, which stands for "tails" on the first toss and "heads" on the second toss.

a. List the four possible results if a coin is tossed two times.

b. List all possible results if a coin is tossed three times. (Hint: One of the possibilities is HTT, which stands for "heads" on the first toss, "tails" on the second toss, and "tails" on the third toss.)

c. If a coin is tossed four times, one possible result is $THHT$. What does $THHT$ stand for? List all possible results that can be obtained by tossing a coin four times.

d. Summarize the results you have obtained thus far by completing the table below.

Number of Tosses of a Coin	Number of Possible Results
1	2
2	4
3	—
4	—

e. Try stating a rule that relates the number of tosses of a coin with the number of possible results. Use the rule you stated to find the number of results that are possible if you toss a coin five times; six times.

The Snowflake

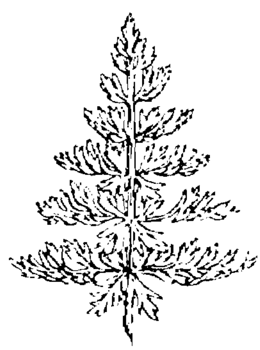
One winter morning Marlene and Mary Kay were chatting while waiting for the school bus. "The snow that is falling makes everything look beautiful," said Mary Kay. "A single snowflake is very beautiful. Did you ever see one up close?" She held out her blue mitten to catch one.

"It is pretty," said Marlene as she inspected the snowflake closely.

"That's because it is symmetrical," replied Mary Kay. Just then the bus came and they got on.

"What do you mean by symmetrical?" asked Marlene.

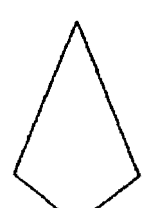
Mary Kay replied that any flat object is symmetrical if a drawing of it can be folded along a line so that the part on one side of the line just fits over the part on the other side.



Caraway leaf

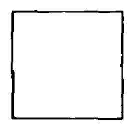
Is this picture symmetrical with respect to line?

Class Discussion 4



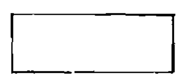
1. Do you think the figure at the right is symmetrical?
   a. Make a tracing of the figure and fold the tracing along a line.
   b. Along what line did you fold the tracing to determine whether or not the given figure is symmetrical?
2. If a figure is symmetrical with respect to a line, we refer to the line as a *line of symmetry*.
   a. How many different lines of symmetry does a square have?

b. Make a copy of the square at the right and sketch the lines of symmetry.



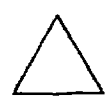
3. a. How many lines of symmetry does a rectangle have?

b. Make a copy of the rectangle at the right and sketch the lines of symmetry.

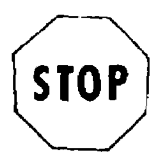


4. a. How many lines of symmetry does an equilateral triangle have?

b. Make a copy of the equilateral triangle at the right and sketch the lines of symmetry.



5. How many lines of symmetry does a highway stop sign have if the word "STOP" is ignored?



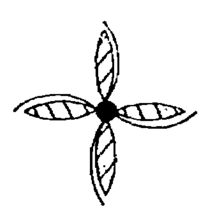
6. How many lines of symmetry do you think a circle has?

7. Mary Kay's sister has a pinwheel like the one shown at the right.

a. Does it appear to be symmetrical in some way?

b. Does it have any lines of symmetry?

c. Can you think of a way to describe this kind of symmetry?



Summary—4

Mary Kay used the language of mathematics to describe the beauty of an object of nature, the snowflake. In the sections that follow you will see many ways in which mathematics can help you think and speak clearly about things you see around you every day.

Exercises—4

1. Bob and Otto, two high school musicians, wanted to form a musical combo. Choosing a name for the combo presented a problem. Mr. Henry, the mathematics teacher, suggested that they choose the name "The Symmetrics." When the boys asked why, Mr. Henry explained, "Because the capital letters used to print your names are symmetrical."

   a. Print the names BOB and OTTO on a sheet of paper. Use capital letters.
   b. Draw a line of symmetry for each name.
   c. For which name is the line of symmetry vertical? For which name is the line of symmetry horizontal?

2. Bob thought Mr. Henry's idea was a good one. "Let's include Ed in the group," said Bob. His name is symmetrical too." Show that the name Ed printed in capital letters is symmetrical.

3. a. Print all capital letters of the alphabet that have a line of symmetry.
   b. Some capital letters have more than one line of symmetry. List them. H is one such letter.

4. Print all digits that have a line of symmetry. Does any digit have more than one line of symmetry? Draw the lines of symmetry in each case.

5. Otto said, "Let's include Anna in our combo. She is a good singer, and besides, her name is symmetrical, too."

"Oh, no, it isn't," said Mr. Henry. "Print ANNA and draw a vertical line between the two N's. Do you see that the A's are symmetrical with respect to the vertical line, but the N's are not?"

"Now I do," said Otto, "but the letter N has some sort of symmetry of its own, doesn't it?"

"Yes, it has what is known as point symmetry. If you locate a point on the letter N as shown at the right, the parts on the two sides of the point have the same shape. But you can't get one part on top of the other by folding along a line. You have to imagine turning one part round and round to get it to fit on the other part."

"Say, I can think of a couple of other letters that have point symmetry," said Otto.

a. Print all capital letters that have point symmetry. You know that N is one such letter. So is S.

b. Locate the point of symmetry for each letter you printed.

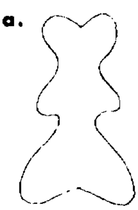
6. Print all digits that have point symmetry.

7. "Some figures that have line symmetry also have point symmetry," said Mr. Henry. "The capital letter H is one example; X is another." Print other capital letters that have both line symmetry and point symmetry.

8. Which of the figures displayed in the exercises of Class Discussion 4 have point symmetry?

9. Which of the figures shown on the opposite page have point symmetry? Which ones have line symmetry? Which have both?

a.



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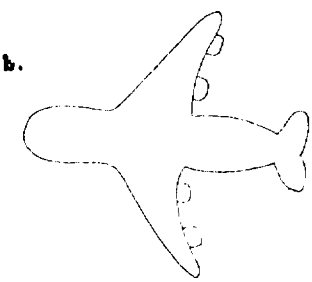

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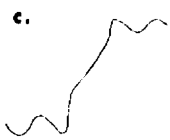

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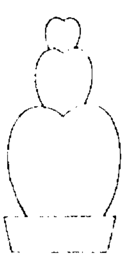
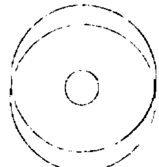
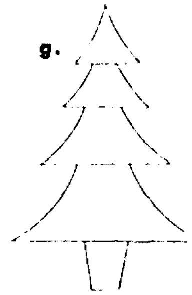


f.



g.



h.

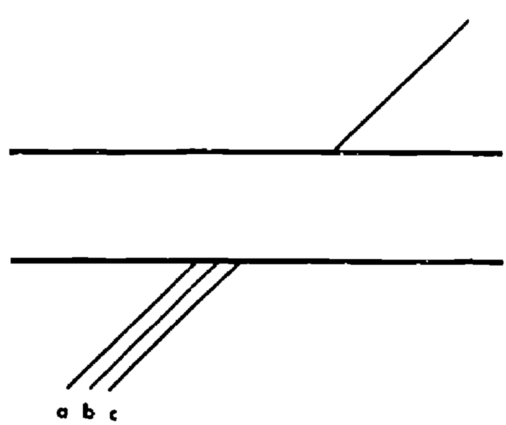
5 Is Seeing Really Believing?

Ev ry day we receive thousands of bits and pieces of information. These come to us through our five senses—sight, hearing, smell, taste, and touch. Our minds sort the information we receive and try to make sense of it all. If we make good use of our senses and think clearly, the conclusions we form will be useful.

Class Discussion 5

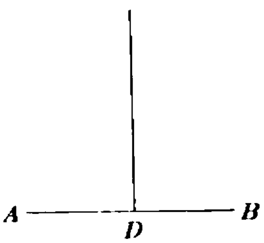
Because seeing is perhaps the most used way of getting information, let us begin by checking how well we make observations. Unfortunately, we do not always see what we think we see.

1. Look at the diagram below. Which segment below is in the same line as the segment above?

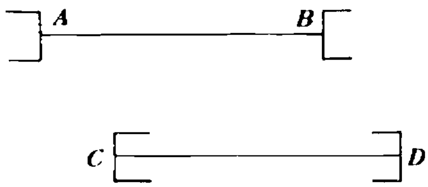


2. In each exercise, look at the figure and decide which segment is longer, $\overline{AB}$ or $\overline{CD}$.

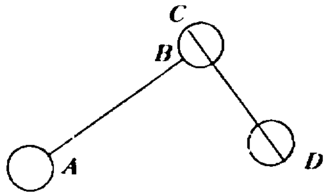
a.



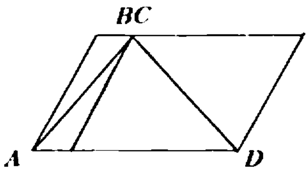
b.



c.

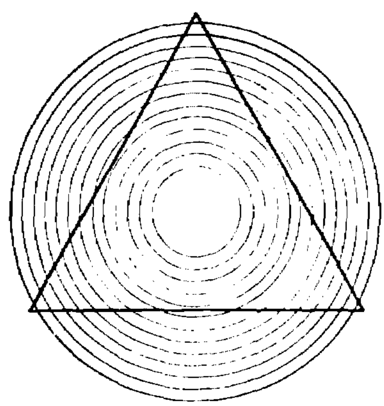


d.

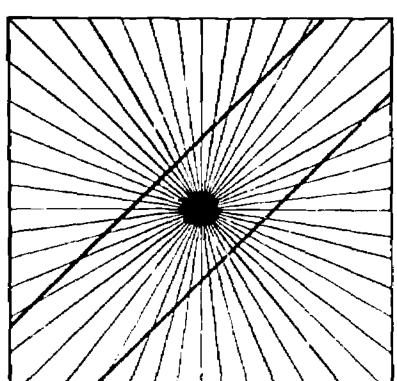


3. Are the long lines parallel in the figure below?

4. Are the sides of the triangle bent or straight?



5. Are the two heavy lines parallel?



Summary—5

If you checked your guesses by using a straightedge or a ruler, you probably found out that things are not always what they "look" like. The above exercises were not included to discourage you from believing your eyes, but rather to encourage you to check the correctness of your observations. After all, if the observations you make are incorrect, then even the clearest thinking can lead to false conclusions.

6 Forming a Generalization—Inductive Thinking

To arrive at useful conclusions, it is important to know how to form a generalization. How does one do this? Here is a pattern of thinking that will help you. Begin by observing specific cases related to the same idea. By observing these specific cases, hopefully a rule that seems true for *all* cases will suggest itself to you. If not, you should "play a hunch" and guess a rule. Such a rule is called a *generalization*. Finally, check to see if the rule you guessed does seem to be true for all cases. This pattern of thinking is called *inductive thinking*.

Class Discussion 6

1. a. The left-hand column of the table below contains a list of counting numbers. If you add the digits of a number in the left-hand column you get the number in the right-hand column. For example, $9 + 8 = 17$. Copy and complete the table.

Number	Sum of Digits of the Number
98	17
225	9
140	5
198	18
200	
1206	
461	
3915	

b. Circle each number in the left-hand column that is divisible by 9. (A counting number is divisible by 9 if the quotient obtained when you divide by 9 is another counting number and there is no remainder.)

c. Check the "Sum of Digits" column in the table and circle each sum that is divisible by 9.

d. Look for a relationship between the numbers in the table that are divisible by 9 and the sums of digits that are divisible by 9.

e. State a rule that relates counting numbers that are divisible by 9 and the sums of the digits of these numbers.

f. What is a foolproof way of obtaining counting numbers that are divisible by 9?

g. Now check to see if the rule you stated in exercise 1e seems to hold for all counting numbers. Make two tables with the same headings as the table above. In the first table, list six counting numbers that you know are divisible by 9. Don't pick small numbers, but get some in the hundreds and thousands. Then find the sums of the digits of these numbers.

In the second table, list six counting numbers that you know for sure are *not* divisible by 9. Find the sums of the digits of these numbers. If the rule you stated in exercise 1e is a good one, it will hold without exception for the results in the two tables you just made.

2. In this exercise you are asked to find a rule for getting the sum of the first *n* odd counting numbers.

First *n* Odd Counting Numbers	*n*	Sum
1	1	1
1, 3	2	4
1, 3, 5	3	9
1, 3, 5, 7	4	16
1, 3, 5, 7, 9	5	
1, 3, 5, 7, 9, 11	6	
1, 3, 5, 7, 9, 11, 13	7	

a. Copy the table.

b. In the table, n stands for the number of consecutive odd counting numbers that are added. The list in each row begins with 1. Complete the table.

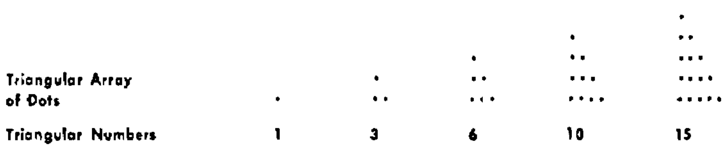
c. Try to state a rule for finding the sum of the first n odd counting numbers by examining the relationship between the numbers in the "n" column and the entries in the "Sum" column.

d. Check the rule you stated for $n = 8, 9, 10, 11$.

e. Do you think your rule will hold for any number n?

f. Do you think these few cases *prove* that your rule will work for every number n?

3. This exercise deals with an interesting pattern that exists among certain counting numbers called triangular numbers. A counting number is a triangular number if it is the number of a set of dots that can be put in a triangular array as shown below. The diagram shows the first few triangular numbers.



a. What are the next two triangular numbers after 15?

b. Continue the pattern started below.

$$1 = 1$$
$$3 = 1 + 2$$
$$6 = 1 + 2 + 3$$
$$10 = 1 + 2 + 3 + ?$$
$$15 = ?$$
$$? = ?$$
$$? = ?$$

c. Do your answers to exercise 3a fit the pattern started in exercise 3b?

d. Which numbers in the list below are triangular numbers?

56, 66, 84, 91, 100, 102, 105, 120

4. This is a continuation of exercise 3. You are asked to find a formula for the sum of the first n counting numbers. In exercise 2 you had to think only about the sum of the first n *odd* counting numbers. Here you are asked to consider *all* of the first n counting numbers.

   a. Copy and complete the table below. How does this table differ from the one in exercise 2?

First n Counting Numbers	n	Sum
1	1	1
1, 2	2	3
1, 2, 3	3	6
1, 2, 3, 4	4	10
1, 2, 3, 4, 5	5	15
1, 2, 3, 4, 5, 6	6	
1, 2, 3, 4, 5, 6, 7	7	
1, 2, 3, 4, 5, 6, 7, 8	8	
1, 2, 3, 4, 5, 6, 7, 8, 9	9	

   b. Double each number in the "Sum" column. Put the number you get in a new column at the right of the table and label the column "Double of Sum."

   c. Make another column and label it "Product." Find a way to write each number in the "Double of Sum" column as a product of two consecutive counting numbers. For example, $30 = 5 \times 6$, and 5 and 6 are consecutive counting numbers.

   d. How are the two numbers of each "Product" pair related to the number n in the same row?

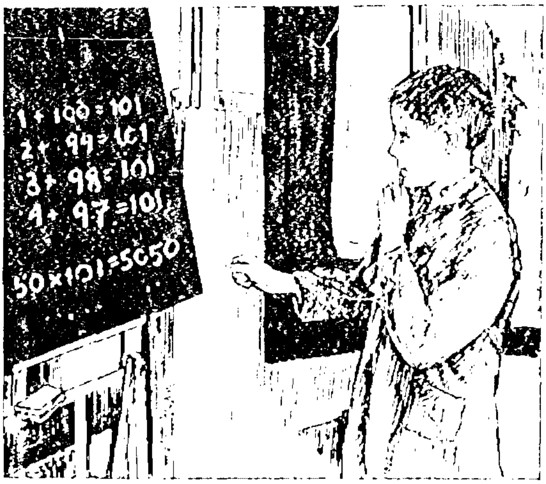
   e. Is $n \times (n + 1)$ equal to the "Double of Sum" in each row?

   f. Write a formula for the "Double of Sum" in each row.

   g. Now write a formula for the "Sum" in each row.

   h. Check the formula you just wrote to see if it holds for each "Sum" in the table.

i. Check to see if the formula you wrote holds for $n = 10$. (Add the first ten counting numbers to see if you get the same results as you do by using the formula.)



Karl Gauss

5. This exercise is closely related to exercise 4. In fact, you are again asked to find a formula for the sum of the first n counting numbers, but in a different way. The questions and discussion are intended to help you see how Karl Gauss, a famous mathematician, discovered a method of finding the sum of the first n counting numbers. Let's see if you can follow his method. (He discovered it while he was still in grade school.)

a. Find the sum of the first six counting numbers.

$$1 + 2 + 3 + 4 + 5 + 6 = ?$$

Check your result by adding "backwards."

$$6 + 5 + 4 + 3 + 2 + 1 = ?$$

Did you get the same sum in both cases?

b. Now get the sum of the first six counting numbers twice, once by adding "frontwards" and once by adding "backwards."

$$(1 + 2 + 3 + 4 + 5 + 6) + (6 + 5 + 4 + 3 + 2 + 1) = ?$$

Is the result twice as large as the sum of the first six counting numbers? If you have a number that is twice as large as you want, how can you get the number you want?

c. In the first line below the indicated sum is written "frontwards." In the second line it is written "backwards." The sum of the numbers in each line is represented by S.

$$S = 1 + 2 + 3 + 4 + 5 + 6.$$
$$S = 6 + 5 + 4 + 3 + 2 + 1.$$

Look at the pairs of numbers that are lined up vertically. What is the sum of each pair? Are all these sums the same? How many vertical pairs are there? What is the product of the *number of pairs* times the *sum of each pair*? Is this product equal to *twice* the sum of the first six counting numbers? What is the sum of the first six counting numbers?

d. Use the same scheme to find the sum of the first eight counting numbers. Begin by writing the indicated sum "frontwards" and "backwards."

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8.$$
$$S = 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1.$$

What is the sum of each vertical pair? How many vertical pairs are there? What is the product of the number of pairs times the sum of each pair? Is this product equal to twice the sum of the first eight counting numbers? What is the sum of the first eight counting numbers?

e. Use the scheme you have been using to find the sum of the first nine counting numbers. Write:

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9.$$
$$S = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1.$$

What is the sum of each vertical pair? How many vertical pairs are there? What is the product of the number of pairs times the sum of each pair? How is this product related to the sum of the first nine counting numbers? What is the sum of the first nine counting numbers?

f. Is there a pattern in the scheme you have been using? When you wanted to find the sum of the first six counting numbers, how many vertical pairs were there? How many vertical pairs were there for the first eight counting numbers? If you wanted to find the sum of the first eleven counting numbers, how many vertical pairs would there be? How many vertical pairs would there be for the first fourteen counting numbers; the first nineteen counting numbers; the first twenty-five counting numbers; the first n counting numbers?

g. Concentrate on the sum of each vertical pair. When you worked with the first six counting numbers, what was the sum of each vertical pair? What was the sum of each vertical pair when you worked with the first eight counting numbers? What was the sum of each vertical pair when you worked with the first nine counting numbers; the first eleven counting numbers; the first fourteen; the first nineteen; the first twenty-five; the first n?

h. In the three examples you worked, was the product of the number of pairs times the sum of each pair always twice as large as the required sum?

i. Try writing a formula for the sum of the first n counting numbers. Use S to represent the sum. If you cannot write such a formula but can explain in words how to find the sum, that's good enough.

j. If you were able to write a formula, using S to represent the sum and n to represent any number you want, then you have done very well and should be congratulated. Does your rule, either as a formula, or in words, agree with the answer you got in exercise 4 above? Are you convinced that your rule can be used to find the sum of the first n counting numbers, no matter what number n represents? Which method of finding the rule was easier, the method of exercise 4 or the method developed in exercises 5a through 5i?

k. Use the rule you have discovered to find the sum of the first eleven counting numbers; the first fourteen; the first nineteen; the first twenty-five; the first one hundred. Trust the rule, rather than adding all numbers each time.

6. A famous mathematician named Goldbach made this conjecture (guess): *Every even number greater than 4 is the sum of two odd prime numbers.* Goldbach knew that his guess was true for a great many numbers, but he could not prove it for all cases. In fact, no mathematician has ever been able to prove Goldbach's guess either true or false. So, to this day, it remains as "Goldbach's Conjecture."

Goldbach: "Even = prime + prime?"

a. A prime number is a counting number that is greater than 1 and is divisible only by itself and 1. Continue the list below until you have listed all prime numbers less than 100.

$$2, 3, 5, 7, 11, 13, 17, \cdots$$

b. The examples below show that Goldbach's Conjecture is true for the first few even numbers.

$6 = 3 + 3.$	$14 = 7 + 7.$
$8 = 5 + 3.$	$14 = 3 + 11.$
$10 = 5 + 5.$	$16 = 11 + 5.$
$12 = 5 + 7.$	$16 = 3 + 13.$

The second column shows that there may be more than one way of expressing an even number as the sum of two odd prime numbers. Goldbach says there is *at least one way* of doing it. Express every even counting number less than 100 as the sum of two odd prime numbers.

c. Did you find other even numbers that can be expressed as the sum of two odd prime numbers in two different ways? List those you found.

d. Are there even numbers that can be expressed as the sum of two odd prime numbers in three different ways? If so, give one example.

Summary—6

One way of arriving at a generalization is by the method of inductive thinking. The method involves this pattern. First you observe specific cases related to some idea. Next you look for a rule that will fit all cases. Then you check the rule in cases that are similar to the first ones you looked at to see if the rule also fits these cases. Generalizations formed in this way are not always true, but they are true rather frequently.

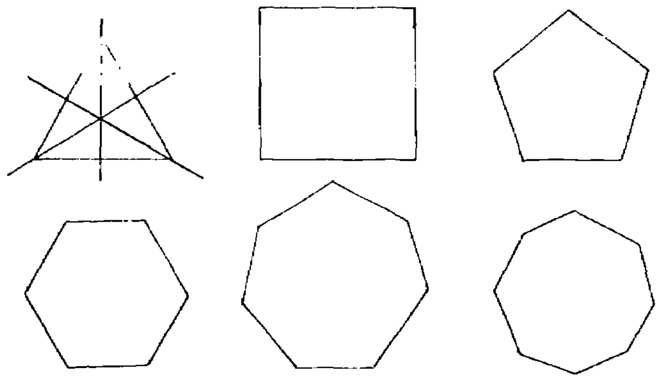
Exercises—6

The first step in inductive thinking is making observations of specific cases. Too often this simply means "looking at" specific cases. You already know that we do not always see what we think we see!

Observations can be sharpened by counting, measuring, or doing experiments. Experiments may involve counting, measuring, or both. Recall the experiments carried out by Sam and Steve (see page 5). On the basis of the measurements they made in their experiments they arrived at this generalization: "The tennis ball we were using is no good."

To do the exercises that follow you will need a ruler and a protractor.

1. This exercise involves getting a rule that can be used to find the number of lines of symmetry of any regular polygon. If you have forgotten what is meant by "line of symmetry," see pages 12-17.



a. Six regular polygons are shown above. A regular polygon with three sides is called an equilateral triangle. An equilateral triangle has three lines of symmetry. Complete the table below for the other polygons shown above.

Number of Sides of the Regular Polygon	3	4	5	6	7	8	9	10	11	12
Number of Lines of Symmetry	3									

b. Complete the entries in the table for regular polygons with 9 sides; 10 sides; 11 sides; 12 sides.

c. Is there a pattern in the completed table? Does the pattern suggest a rule for finding the number of lines of symmetry if you know the number of sides? If so, state the rule. Use N to represent the number of sides and S to represent the number of lines of symmetry.

d. Did you get your answers for polygons with an even number of sides differently than your answers for polygons with an odd number of sides?

e. Are you convinced that your rule will work for any regular polygon? Why, or why not?

2. The object of this exercise is to find a rule for computing the number of diagonals of a polygon if you know the number of sides. The polygon need not be regular. A diagonal is a line segment joining two vertices that are not endpoints of the same side.

a. Complete the table.

Number of Sides of the Polygon	3	4	5	6	7	8
Number of Diagonals	0	2				

b. Is there a pattern in the completed table? Try to describe the pattern.

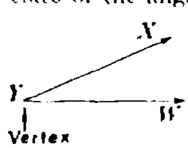
c. Does the pattern suggest a relationship between the number of sides of a polygon and the number of diagonals of the polygon?

d. Give a formula for the number of diagonals of a polygon in terms of the number of sides. (This is not easy!)

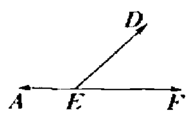
e. Are you convinced that the formula you gave can be used to compute the number of diagonals of any polygon? Why, or why not?

An angle consists of two rays that have a common endpoint but are not in the same line. The common endpoint of the rays is called the *vertex* of the angle. The rays are called the *sides* of the angle.

An angle may be named in various ways. One way is to use three letters and to write the letter that labels the vertex between the other two. Using this scheme, the angle pictured at the right can be named either "angle XYW" or "angle WYX."

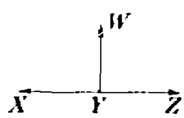


3. Angles AED and DEF have these properties:
   (1) They have a common vertex.
   (2) They have a common side.
   (3) The other two sides form a straight line.

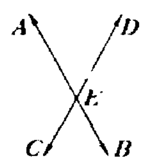


Two angles like AED and DEF are called a linear pair of angles.

   a. Measure angles AED and DEF with a protractor.
   b. Find the sum of the measures of angles AED and DEF.

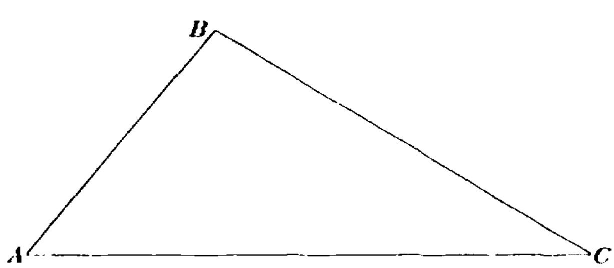


   c. Are angles XYW and ZYW a linear pair of angles? Find the sum of the measures of the two angles.
   d. Name four linear pairs of angles in the figure at the right. Find the sum of the measures of each pair.

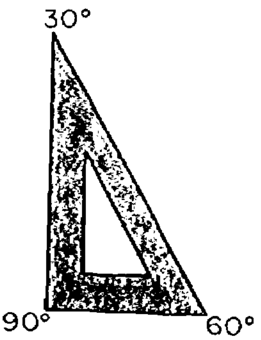


   e. Do your results suggest that the sum of the measures of a linear pair of angles is the same for all such pairs of angles?

4. a. Draw five or more triangles that have different shapes.
   b. Measure the three angles of each triangle with a protractor.
   c. Find the sum of the measures of the angles of each triangle.
   d. Do your results suggest that the sum of the measures of the three angles of all triangles is the same?
   e. Without measuring the angles of triangle ABC, guess what you think the sum of the measures is. Now measure the angles and find the sum of the measures.



5. Consider a triangle that has two sides that are the same length. Such a triangle is called an isosceles triangle.
   a. Draw at least five isosceles triangles that have different shapes. Remember that two sides in each triangle must have the same length. The angle formed by the two sides that have the same length is called the vertex angle of the isosceles triangle.
   b. In each triangle, measure the two angles that are opposite the two sides that have the same length. These two angles are called the base angles of the isosceles triangle.
   c. How do you think the measures of the two base angles of an isosceles triangle are related?



6. A draftsman uses a plastic triangle with angles that have measures of 30°, 60°, and 90°. A triangle like this is often called a "30°-60°-90° right triangle." Any angle that has a measure of 90° is called a right angle, and a triangle with a right angle is called a right triangle. The side opposite the right angle in a right triangle is called the hypotenuse.
   a. Use a protractor to draw five 30°-60°-90° right triangles. Make each triangle a different size.
   b. Copy and complete the table below by measuring the sides whose lengths are called for in the table. If possible, use a

30°-60°-90° Right Triangles	Length of the Side Opposite the 30° Angle	Length of the Hypotenuse
I		
II		
III		
IV		
V		

ruler with a metric scale. Record measurements to the nearest millimeter.

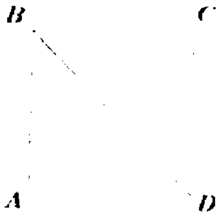
c. Is there a relationship between the length of the hypotenuse and the length of the side opposite the 30° angle? Express this relationship in words.

d. Do you think the relationship you expressed is true for all 30°-60°-90° right triangles?

e. Without using a ruler, give the length of the hypotenuse of a 30°-60°-90° right triangle if the length of the side opposite the 30° angle is 60 millimeters; 5 inches; 31 feet; 101 millimeters; 19 yards; 111 meters.

f. Without using a ruler, give the length of the side opposite the 30° angle in a 30°-60°-90° right triangle if the length of the hypotenuse is 52 centimeters; 4 inches; 14 feet; 13 meters; 3 feet, 8 inches.

7. If you are given the length of the side of a square, how would you go about estimating the length of a diagonal of the square? Let's try to find out. Pictured at the right is square *ABCD* with diagonal *BD*. Notice that the square and the diagonal form two isosceles triangles.



a. Is each isosceles triangle also a right triangle? What are the measures of the base angles of each isosceles right triangle? (Try to answer this question without measuring.) Is diagonal *BD* of the square the hypotenuse of each isosceles right triangle? Do the two isosceles right triangles have the same shape and size? To find the length of the diagonal of a square when the length of a side is given, you need to look at only one of the isosceles right triangles.

b. Draw five different squares. No two should be the same size. Choose a whole number of units (for example 3 inches, 5 inches, 4 millimeters) for the length of a side in each. Label the squares I, II, III, IV, and V.

c. Copy the table below and record the length of a side for each square.

Square	Length of a Side	Length of a Diagonal	$\frac{\text{Length of a Diagonal}}{\text{Length of a Side}}$
I			
II			
III			
IV			
V			

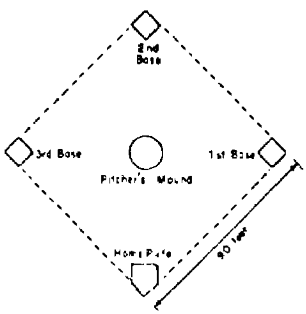
d. Measure the length of a diagonal in each square using the same unit that you used to measure the side. Record the measurements in the table.

e. For each square, divide the length of a diagonal by the length of a side. Find the quotient to the nearest thousandth. Record your results in the table.

f. The five quotients that you got should all be approximately equal to the same number. To get the best estimate of what this number should be, find the average of the five quotients. Compare the average you get with that of your classmates.

g. Take the average you got and multiply it by itself. Round off the product to the nearest hundredth. Is your answer close to some whole number?

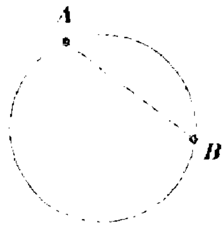
BASEBALL DIAMOND



h. Explain how to estimate the length of a diagonal of a square when you are given the length of a side.

i. Estimate the distance from home plate to second base on a baseball diamond. (The distance from home plate to first base is 90 feet.)

8. a. Draw a circle on a sheet of paper.
   b. Choose two different points on the circle and label them A and B.
   c. Connect A and B with a segment. A segment with its endpoints on a circle is called a *chord.*
   d. Into how many separate subregions does chord AB divide the interior of the circle?



   e. Select a third point on the circle and label it C.
   f. Draw chords AC and BC.
   g. Into how many separate subregions do chords AB, AC, and BC divide the interior of the circle? (In counting subregions do not count any part more than once.)
   h. Select a fourth point and label it D.
   i. Draw chords AD, BD, and CD.
   j. Into how many separate subregions is the interior of the circle divided now?
   k. Copy and complete the table below using the answers you gave in exercises 8d, 8g, and 8j.

Number of Points on the Circle	Number of Separate Subregions in the Interior of the Circle
2	
3	
4	

   l. Make a guess as to the number of separate subregions you would get if you drew chords connecting five points on the circle. Check your guess by selecting five points, drawing the required chords, and counting the subregions you get.
   m. Do your results seem to follow a pattern? Do you see a relationship between the number of points and the number of subregions? If so, state a rule that expresses this relationship. Do you think your rule is true for any number of points on the circle?

n. Use your rule to find the number of subregions there would be if you selected six points on the circle.

o. Check your answer to the last exercise by drawing the required connecting chords and counting subregions. Was your answer to the last exercise correct?

p. If you completed exercises 8a through 8o you should have discovered one of the pitfalls of inductive thinking. What is this pitfall? What can you do to avoid it?

9. This exercise is a real challenge. If you don't see the pattern, don't feel discouraged.

a. Draw a large circle on a sheet of paper. Choose points A and B on the circle and connect them by drawing chord AB. Into how many subregions does AB divide the interior of the circle? (Caution: If your answer isn't "two," stop here.)

b. Draw a second chord CD, that intersects chord AB at some point inside the circle. How many subregions are there now?

c. If you are wondering what's so hard about this problem, consider this: Draw a third chord, EF, that intersects each of the chords already drawn at points where there are no other intersections. Count the subregions as before.

d. At this point, before you draw any more chords or do any more counting of subregions, guess the number of subregions that would be formed by four intersecting chords; by five.

e. Draw a fourth chord, GH, that intersects each of chords AB, CD, and EF inside the circle at points where there are no other intersections. Count the number of subregions that are formed. Does your count agree with the guess you made?

f. Copy and complete the table below.

Number of Chords	Number of Subregions
1	2
2	4
3	
4	
5	

g. Guess the number of subregions for six chords. Check your guess by making a drawing. Is your guess correct?

h. Write a formula that fits all cases examined thus far. Your formula should give you the number of subregions S in terms of the number of chords C.

7 Thinking by Analogy

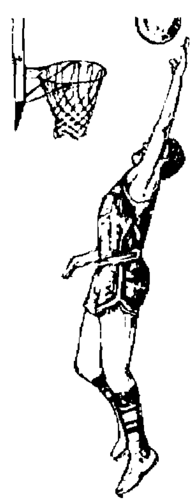
Mr. Wilson, the basketball coach of Canyon High School, was talking to his assistant coach, Mr. Zientek. Mr. Wilson said, "I think Willie Gray will be a good basketball player."

"Why do you think so?", asked Mr. Zientek, "You've never seen him play!"

"Well," said the coach, "Willie is 6 feet 4 inches tall and is a good football player. Ron Davis, who is also 6 feet 4 inches tall, is both a good football player and a good basketball player. Therefore, I think Willie will be a good basketball player, too."

The kind of thinking Mr. Wilson did is called *thinking by analogy*. A person is said to think by analogy if his thinking follows this pattern. He observes that two persons or things are alike in one or more ways. On the basis of this information he concludes that maybe the two persons or things are alike in other ways.

Class Discussion 7



1. Do you think Coach Wilson's hunch about Willie's ability to play basketball is reasonable? How could Coach Wilson check his hunch?
2. Suppose Willie Gray turns out to be a good basketball player, and a good chess player besides. If you think by analogy, what would you conclude about Ron? Why is it more sensible to think by analogy in exercise 1 than in this exercise?
3. Look again at exercise 1 on pages 21-22.
   a. If you completed this exercise you should have discovered the following rule for testing divisibility by 9: A counting number is divisible by 9 if and only if the sum of its digits is divisible by 9.
   b. Now look for a rule to test divisibility by 3. This rule should be similar to the one for testing divisibility by 9. First of all, the new rule is concerned with divisibility. Secondly, it deals with the number 3, which is a factor of 9. State a rule for testing divisibility by 3.
   c. Check the rule you just stated. Listed below are five numbers that are divisible by 3.

      12, 42, 87, 192, 2316

      Does your rule provide you with a test?
   d. Listed below are five numbers that are not divisible by 3. See if your rule serves as a test.

      17, 58, 683, 788, 1072

e. If the rule you stated for testing divisibility by 3 is correct, it should work with the five numbers listed below. See if they are divisible by 3, first by dividing, then by using your rule.

10101, 216, 76, 99192, 4618

f. Do you think your rule for testing divisibility by 3 will always work?

4. As you have seen, the rules for testing divisibility by 3 and 9 are *analogous*. Check to see if there is a similar rule for testing divisibility by 6. The number 6 seems to be in the same "family" as 3 and 9. State a rule for testing divisibility by 6 that is analogous to the rules for testing divisibility by 3 and 9.
   a. Check the rule you stated.
   b. Do you see that the rule for testing divisibility by 6 that is analogous to the rules for testing divisibility by 3 and 9 does *not* work? If you think it does, use your rule to test whether 36 and 123 are divisible by 6.
   c. State a rule for testing divisibility by 6 that *does* work.

5. State rules for testing divisibility by 2, 4, 5, 7, or 8 that are analogous to the rules for testing divisibility by 3 and 9. Do these rules work?

Summary—7

Thinking by analogy does not guarantee that conclusions you form will be true. However, thinking by analogy is a good way to discover facts or generalizations that may be true.

Exercises—7

Each exercise describes a situation that involves thinking by analogy. In each exercise tell whether you think the people have used clear thinking and if the conclusions they formed are likely to be true. If you don't think the conclusions are true, tell why.


Bonnie

1. Maggie is young and good-looking but has difficulty making friends at the ski lodge. On TV she saw an advertisement showing a young girl, Bonnie, who was good-looking and who was also having trouble making friends at the ski lodge. Bonnie was told by a friend to use "Sweet Smile" mouthwash. Sure enough, weeks later Bonnie was being showered with attention by a very handsome man at the ski lodge. Maggie decided that she should use "Sweet Smile" too.

2. Jennifer got high marks in grade school and is also getting high marks in high school. Louise, Jennifer's younger sister, is getting about the same marks in grade school that Jennifer did. The girls' mother thinks Louise will get high marks in high school.

3. Last December, Mr. Walters sold an average of $750 worth of clothing per day. He figures he will average about $750 per day in clothing sales this coming December.

4. Chet told Bill that he has averaged 17.3 miles per gallon with his 1961 Plymouth V-8 since he switched to "Super Blue" gasoline. Bill, who also has a 1961 Plymouth V-8, has been getting only 15.8 miles per gallon. But Bill has been using "Uncle Jake's" economy gasoline. Bill decides to switch to "Super Blue" gasoline too.

5. Mrs. Thompson told her husband, "Let's not plan on taking the children to the Thanksgiving Day parade this year. For the past three years it has been cold and rainy on Thanksgiving Day."

6. Toni Taylor, the movie star, looks very attractive in a red dress. Darlene, who has black hair and hazel eyes, just like Toni Taylor, thinks she should wear a red dress, too.

7. Mr. Arnold Glover is a popular and successful professional golfer.

Mr. Glover says his success is the result of many hours of practice as a youngster and eating "Wheatos" for breakfast every day. Randy would like to become a professional golfer. So Randy decides to start eating "Wheatos" for breakfast every day.

8 Deciding whether a Statement Is True or False

You have used various ways of forming a generalization. But none of the ways you used guaranteed that the generalizations you formed would be true. An important step in clear thinking is checking whether or not a generalization is true. In this section you will explore ways of doing this.

Class Discussion 8a

To decide whether a statement is true or false, first make sure you understand the meanings of the words and symbols that are used in the statement.

1. For which of the following statements is it difficult or impossible for you to decide whether the statement is true or false? In each case, tell what causes the difficulty.
   - **a.** 342 is divisible by 6.
   - **b.** If you are patriotic, then you will vote on election day.
   - **c.** He is the most popular boy in school.
   - **d.** The set of prime numbers less than 100 contains 22 numbers.
   - **e.** A formula for finding the sum of the first n counting numbers is

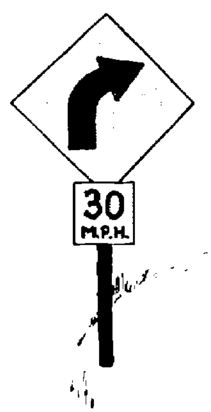
$$S = \frac{n \times (n + 1)}{2}.$$

   - **f.** A supercalifragillistic bojohn is an expiallidocious frump.

g. Driving at a speed of 30 miles per hour is dangerous.
h. The square root of 20 is not a whole number.
i. All policemen are brave.
j. Cheating on examinations is dishonest.
k. All snigs are wiggs.
l. The measure of every angle of a pentagon is less than 90°.
m. Butter is better.
n. Mathematics is fun.

2. Which statements in exercise 1 were easy for you to judge true or false? Tell how you would go about making a check in each case.

WARNING!



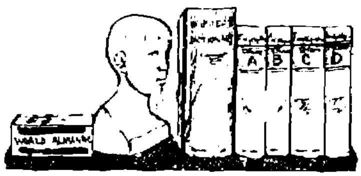
Class Discussion 8b

There are various ways of checking the truth of a statement that deals with a specific situation.

Suppose that a visitor comes in from outdoors and says, "It is raining." A convincing way to check the truth of the visitor's statement is to look out the window and make a *direct observation*. Unfortunately, it is not always possible to check the truth of a statement directly. If there are no windows, you might use indirect evidence, such as the fact that the visitor is dripping wet. However, if the visitor kept himself dry with an umbrella, you might have to rely on his reputation for honesty and accept his word. The method you use to check the truth of a statement determines how sure you can be of your decision.

1. For each statement, tell what sort of observation, test, or check you would make to decide whether the statement is true or false.

a. Bill can throw a football farther than Wally.
b. The set of all prime numbers less than 50 contains 15 members.
c. 676 is a perfect square.
d. Your history textbook has 347 pages.



e. Mr. Thomas Schneider lives at 147 East 11th St., New York City.
f. A new tennis ball will bounce more than half the height from which it is dropped.

2. Sometimes you can check on the truth of a statement indirectly by talking with someone, or by looking in a book such as a dictionary or an almanac. Doing this is known as *consulting an authority*. For example, if you want to know whether it is true that Babe Ruth hit 52 home runs in 1922, you might consult an official record book of baseball. For each statement below, tell what authority might be consulted to determine whether the statement is true or false.
   a. John Adams was the third president of the United States.
   b. The correct spelling of the word is "paralell."
   c. Mt. Rainier is 13,271 ft. high.
   d. Philadelphia is the third largest city in the United States.
   e. 163 is a prime number.
   f. The name of a twelve-sided polygon is "duodecagon."
   g. Louis XIII was king of France in 1843.
   h. A square is a rhombus.
   i. The capital of Brazil is Rio de Janeiro.

j. The largest body of fresh water in the world is Lake Superior.

3. Suppose you consulted a book dated 1940 to check on the statements in exercises 2d and 2i. What would your decisions be about the truth of these statements?

Class Discussion **8c**

There is one very important way of deciding whether a statement is true or false. This is known as the method of deductive thinking. The method is most often used to decide whether a generalization is true or false. However, it can also be used to decide if a statement dealing with a specific situation is true or false. Using the method of deductive thinking to show that a statement is true (or that it is false) is referred to as *proving* the statement.

Doing the exercises below will help you understand what is meant by deductive thinking.

1. Suppose there are 20 students in your mathematics class. Is the following statement true? At least two students in the mathematics class have birthdays in the same month.

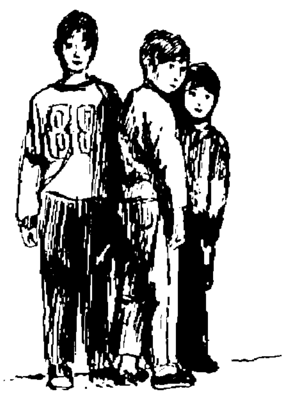
2. Susan lives in Minneapolis. Is it true that Susan lives in the State of Minnesota?

3. If you know that George is an ordinary cat, does it follow that George has four legs?

4. To be a policeman in Los Angeles a man must be taller than 5 feet 6 inches. Mr. Manuel Figuera is a policeman in Los Angeles. What conclusion follows from these facts?

5. Phil's teacher asked this question: "If the vertex angle of an isosceles triangle measures 80°, what is the measure of a base angle of the triangle?" Phil answered, "55°." Do you agree with Phil's answer?

6. If Fred is taller than Pete, and Pete is taller than Jim, what do you know about the heights of Fred and Jim?


Fred, Pete, and Jim

7. Isadore turned on his radio to listen to the New York Mets baseball game. The announcer said "This is the last half of the ninth inning and the Mets are batting." Isadore turned to his father and said, "It looks like the Mets are not ahead." Was Isadore's statement true?

8. Can you form a conclusion about whether your school is having classes today if you know that today is Sunday?

Summary—8c

In each exercise above you were given some information and asked to form a conclusion. You were expected to accept the given information as true. In some exercises enough information was given so that you were able to decide on the truth of the conclusion at once. In others, you had to call on your memory for additional information you believed to be true. More than likely, you did not write down the information you got from your memory. In deductive thinking, a statement (written or unwritten) that is accepted as true for the sake of argument is called an *assumption*. A decision regarding the truth of a conclusion must be based on whatever assumptions you make.

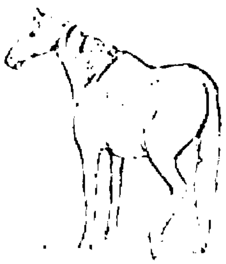
Exercises—8c

1. In which exercise in Class Discussion 8c were you given all the information needed to make a decision? In which exercises did

you need to call on your memory for additional information you believed to be true? What was this information?

2. Assume all statements given in each exercise below are true. State a conclusion for each exercise that follows from the given statements.
   a. Jerry can swim farther than Roger. Roger can swim farther than Henry.
   b. Texas is west of the Mississippi River. Charlene lives in Texas.
   c. Columbus is a city in Ohio. Ohio is in the United States.
   d. All birds have feathers. A robin is a bird.
   e. Candy bars at the corner drugstore cost 10 cents each. Fritz has enough money to buy 2 candy bars but not enough to buy 3 candy bars.
   f. An even number is a number that is divisible by 2. The number 936 is even.
   g. All snigs are wiggs. Bobby is a snig.
   h. If a person is fat, he is happy. Prunella is fat.
   i. If you use Crunch toothpaste, you will have no cavities. Art uses Crunch toothpaste.
   j. A number is divisible by 5 only if the unit's digit is 0 or 5. 473 is a number.

3. In each exercise below accept the given statement as true. The conclusion in each exercise follows from the given statement and from an additional statement or statements that are understood to be true. Tell what the additional statement or statements are.
   a. Given: Paint is a horse.
      Conclusion: Paint has four legs.
   b. Given: If schools are closed, then the students are happy.
      Conclusion: The students are happy today.
   c. Given: The measure of one angle of a triangle is 48° and the measure of a second angle is 58°.

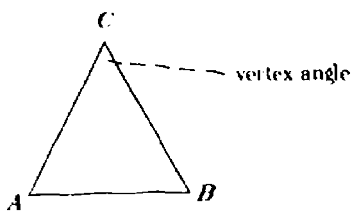


Conclusion: The measure of the third angle of the triangle is 74°.

d. Given: Kay is a student in advanced homemaking.
Conclusion: Kay is required to make three dresses.

e. Given: A dime is a United States coin.
Conclusion: A dime is round.

f. Given: Triangle ABC is isosceles.
The measure of the vertex angle is 50°.
Conclusion: The measure of each base angle is 65°.



g. Given: The number of diagonals of a certain polygon is two.
Conclusion: The polygon has four sides.

Class Discussion 8d

Are you starting to understand what deductive thinking is? You are doing deductive thinking if you base conclusions on statements that you accept as true. If the statements you accept as true make your conclusions necessary, then your deductive thinking is correct.

Now here is a strange thing which you must try to understand. It can happen that a conclusion you form is not true even though the deductive thinking you do is correct. This can happen when you base a conclusion on a statement that is not true. So, if you want your conclusions to be true, you will have to be doubly careful. First, you must be sure that the statements you accept as true really are true. Second, you must be sure that your deductive thinking is correct.

In each exercise concentrate on doing correct deductive thinking.

Accept each given statement as *true*, even though you may not believe that it is. Then tell whether the conclusion follows from the given statements you accepted as true.

1. Given: All policemen are taller than six feet. Bill Thompson is a policeman.
   Conclusion: Bill Thompson is over six feet tall.

2. Given: All giraffes have four feet. All antelopes have four feet.
   Conclusion: All antelopes are giraffes.

3. Given: I. Have Lotsabucks belongs to the Hi-Ritz Country Club. I. Have Lotsabucks is a wealthy person.
   Conclusion: Only wealthy persons belong to the Hi-Ritz Country Club.

4. Given: If you have exactly 17 cents in your pocket, then you have at least four coins in your pocket. You have fewer than four coins in your pocket.
   Conclusion: You do not have exactly 17 cents in your pocket.

5. Given: All prime numbers are odd numbers. 2 is an even number.
   Conclusion: 2 is not a prime number.

6. Given: New York City is in Pennsylvania. Max lives in New York City.
   Conclusion: Max lives in Pennsylvania.

7. Given: If the weather is clear today, we shall have a picnic. We are not having a picnic today.
   Conclusion: The weather is not clear today.

8. Given: If each of two angles is a right angle, then the two angles are the same size. The two angles are not right angles.
   Conclusion: The two given angles are not the same size.

9. Given: All snigs are wiggs. You are not a wigg.
   Conclusion: You are not a snig.

10. Given: If I look at TV for more than an hour, I get a headache. I have a headache.
    Conclusion: I was looking at TV for more than an hour.

11. Given: If a man uses Code 007 hair cream, he will be attractive to girls. Rock Hunter uses Code 007 hair cream.
    Conclusion: Rock Hunter is attractive to girls.

12. Given: Tobias Drinkwater is a guitar player. Tobias Drinkwater drives a Ford convertible.
    Conclusion: Some guitar players drive Ford convertibles.

13. Given: Some high school boys wear long hair and blue jeans. Randy Rucksack is a high school student.
    Conclusion: Randy Rucksack wears long hair and blue jeans.

Randy Rucksack

14. Given: All blondes have fun. Prisoners don't have fun.
    Conclusion: Prisoners are not blondes.

15. **a.** Some conclusions in exercises 1–14 are really false even though they follow from the given statements. For each of these, tell why the conclusion is false.
    b. Some of the conclusions above are hard to judge true or false. For each of these, tell what causes the difficulty.

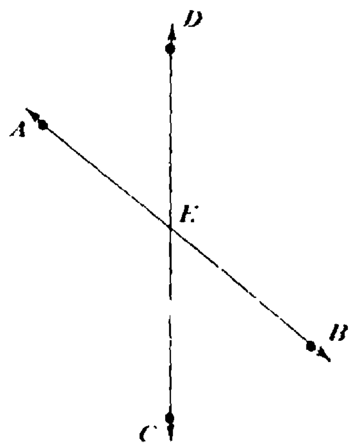
16. Do you think it is possible for true conclusions to follow from false assumptions? Give an example.

Summary—8d

To obtain true conclusions when doing deductive thinking two things are necessary. First of all, the statements on which you base your conclusions must be true. Second, you must use the method of deductive thinking correctly.

Exercises—8d

1. Draw a pair of intersecting lines AB and CD. Label the point of intersection E.



2. Angles like AED and BEC are called opposite angles. Use a protractor to find the measures of angles AED and BEC. How do the two measures compare?
3. Find the measures of angles AEC and BED. How do the two measures compare?
4. **a.** Draw another pair of intersecting lines. Find the measures of one pair of opposite angles. How do the measures compare?
   b. Find the measures of the other pair of opposite angles. How do these two measures compare?
5. State a generalization about the measures of a pair of opposite angles formed by two intersecting lines.

6. Do your results suggest that the measures of a pair of opposite angles formed by two intersecting lines are equal? To prove this is true you need to use the method of deductive thinking.
   a. In exercise 3 on pages 47–48 you probably decided that the sum of the measures of a linear pair of angles is 180°. Assume this is true.
   b. Why is it true that the sum of the measures of angles *AED* and *BED* is 180°?
   c. Why is it true that the sum of the measures of angles *CEB* and *BED* is 180°?
   d. Think about subtracting the measure of angle *BED* from 180°. Will the result be the measure of angle *AED*? Will the result also be the measure of angle *CEB*?
   e. Is the measure of angle *AED* equal to the measure of angle *CEB*?
   f. Does the thinking you did in exercises 6a–6e prove that the measures of a pair of opposite angles formed by two intersecting lines are equal?

General Summary

This unit was written to help you learn how to form useful conclusions by doing clear thinking. Opportunity was provided for you to examine various methods of arriving at conclusions and of deciding whether conclusions were true. In this connection you saw how faulty thinking can lead to false conclusions. For example, using words that mean different things to different people can lead to false conclusions. Making hasty observations, "jumping to conclusions," and doing incorrect deductive thinking can also lead to false conclusions.

Inductive thinking is a method of thinking by which you can arrive at generalizations. The method involves observing specific cases, finding a pattern in the cases observed, and making a guess that the pattern will be the same in all cases that are similar to those observed.

Thinking by analogy means concluding that two specific cases that are alike in some ways may be alike in other ways.

A conclusion reached either by inductive thinking or thinking by analogy should be checked to see if it is true or false.

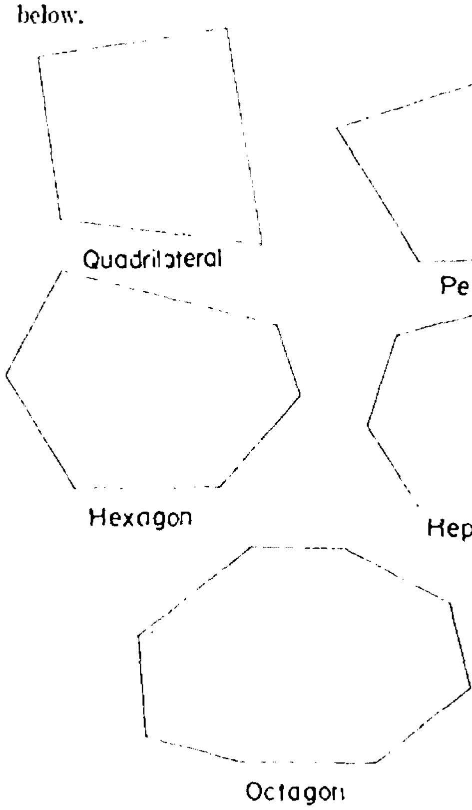
Specific conclusions may be checked by making observations, doing experiments, consulting authority, or by deductive thinking. The truth of a generalization is harder to check. If a generalization holds in case after case, you may "feel" convinced that it is true. But to be sure that it is true, you need to know in some way why it is true. Using the method of deductive thinking is the usual way of determining this. A person who uses the method of deductive thinking begins with accepted statements which are known to be true and forms a conclusion that follows from (or is based on) the accepted statements.

As you read newspapers and magazines, watch television, or simply talk with people, you will find both correct thinking and faulty thinking being used to support conclusions. This unit should help you think correctly in arriving at conclusions, both in mathematics and in daily life situations.

Review Exercises

This set of exercises concerns many of the ideas and techniques that you studied in this unit.

1. In exercise 4 on page 32 you should have arrived at the conclusion that the sum of the measures of the angles of a triangle is 180°. Assume this is true. Then use the inductive thinking method to find the sum of the measures of the angles of polygons with more than three sides. Begin by considering the polygons shown below.



a. Draw two quadrilaterals; two pentagons; two hexagons; two heptagons; two octagons. The two polygons in each pair should have different shapes. Make your drawings large enough so that you can measure the angles in each figure with a protractor.

b. Measure each angle in each of the ten polygons. Find the sum of the measures of the angles in each polygon.

c. Copy and complete the table below.

Polygon		Number of Sides	Sum of the Angle Measures
Quadrilateral	I	4	
Quadrilateral	II	4	
Pentagon	I		
Pentagon	II		
Hexagon	I		
Hexagon	II		
Heptagon	I		
Heptagon	II		
Octagon	I		
Octagon	II		

d. The sum of the measures of the angles for each pair of polygons should be the same. What do you think the sum is for each pair?

e. Do you see a pattern in your results? If so, state a rule for the pattern. Or better yet, write a formula for the pattern.

f. Use the rule you stated to predict what the sum of the measures of the angles of a nine-sided polygon should be. Check your prediction by drawing a nine-sided polygon and measuring the angles.

g. Do the measurements you made *prove* that your rule is correct? Do you believe that your rule is correct?

2. If you followed instructions thus far, you should have arrived at a generalization that you *believe* to be true. To *prove* beyond a shadow of a doubt that it is true, you need to use the method of deductive thinking.

a. Make a copy of each polygon shown in exercise 1. In each polygon choose one vertex and label it *A*.

b. From vertex A, draw as many different diagonals as you can.

c. By drawing as many different diagonals as you can from one vertex, any polygon can be divided into triangles. For example, in the pentagon, exactly two different diagonals can be drawn from one vertex, and there are three triangles. Copy and complete the table below.

Polygon	Number of Sides	Number of Diagonals from Vertex A	Number of Triangles Formed
Quadrilateral			
Pentagon			
Hexagon			
Heptagon			
Octagon			

d. Use the table above to describe the relationship between the number of sides of a polygon and the number of triangles formed by drawing as many different diagonals from one vertex as you can.

e. Do you see that the angles of a polygon are made up of all the angles of all the triangles in a polygon?

f. Assume it is true that the sum of the measures of the angles of a triangle is 180°. What do you think the sum of the measures of the angles of a quadrilateral should be? Why do you think so?

g. What do you think the sum of the measures of the angles of a pentagon should be? Why do you think so?

h. What do you think the sum of the measures of the angles of a hexagon should be? Of a heptagon? Of an octagon?

i. In a polygon with twelve sides, how many diagonals can be drawn from a single vertex? How many triangles would be formed if these diagonals were drawn? What do you think the sum of the measures of the angles of a twelve-sided polygon should be? Why do you think so?

j. Repeat the last exercise for a polygon with 20 sides; 50 sides; 1,000 sides; n sides.